NASA TECHNICAL NOTE

NASA TN D-4763

NASA TN D-4763

AN EFFICIENT METHOD FOR COMPUTATION OF CHARACTER TABLES OF FINITE GROUPS

by Gabriel Allen

Lewis Research Center
Cleveland, Ohio

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • AUGUST 1968

N68-31961

ERRATA

NASA Technical Note D-4763

AN EFFICIENT METHOD FOR COMPUTATION OF CHARACTER TABLES OF FINITE GROUPS

by Gabriel Allen

August 1968

Page 23, Character Table for Group D_2: The fourth column should be labeled R_3^2 instead of R_4^2.

Page 27, Group Table for Group T_d or O: The row labeled g_{22} should read

$|i_4|\ i_5\ \ i_6\ \ R_1^3\ \ R_1\ \ i_2\ \ i_1\ \ R_2^3\ \ R_2|$ . . . instead of $|i_4|\ i_5\ \ i_6\ \ R_1^3\ \ R_1\ \ i_1\ \ i_2\ \ R_2^3\ \ R_2|$ . . .

NASA TN D-4763

AN EFFICIENT METHOD FOR COMPUTATION OF CHARACTER TABLES OF FINITE GROUPS

By Gabriel Allen

Lewis Research Center
Cleveland, Ohio

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – CFSTI price $3.00

ABSTRACT

A systematic procedure is presented for constructing the character table of a given finite group. The use of this procedure makes the task of computing group character tables more straightforward than previously published procedures. Each step in the construction of the character tables is illustrated by worked out examples. An appendix of group tables, character tables, and class algebra tables for many of the common finite groups is included.

AN EFFICIENT METHOD FOR COMPUTATION OF CHARACTER TABLES OF FINITE GROUPS

by Gabriel Allen

Lewis Research Center

SUMMARY

A systematic procedure is presented for constructing the character table of a given finite group. The use of this procedure makes the task of computing group character tables more straightforward than previously published procedures. Each step in the construction of the character tables is illustrated by worked out examples. An appendix of group tables, character tables, and class algebra tables for many of the common finite groups is included.

INTRODUCTION

The reader should be familiar with elementary group theory at least to the extent of knowing the definition of common terms (ref. 1). The application of group theory to physical and chemical problems is now common. Group theory is a systematic and efficient way of exploiting the symmetry in physical systems to avoid duplication of computation. Once the collection of symmetry elements has been identified as a known group (this identification is often tantamount to a geometric exercise), the character table and irreducible representations are the most useful properties of the group.

The determination of character tables (CT) and irreducible representations (IR) are standard topics of elementary group theory (refs. 2 and 3). However, most descriptions of these procedures appear to depend on an intuitive feeling about each particular group. For this reason, a systematic and efficient procedure for constructing character tables of finite groups is presented. The procedure is due to Harter (ref. 4) and does not seem to be well known.

Procedures for computing irreducible representations are also important, but improved methods for finding them require considerable extension of elementary group theory. Such extensions have been made (refs. 4 and 5) but will not be described herein.

Recently, Harter has made additional extensions enabling CT's and IR's of ray algebras to be computed efficiently (unpublished data obtained from W. G. Harter.)

Some common finite groups, their character tables, and class algebra tables are included in an appendix.

PROCEDURE

In a very broad outline, the procedure consists of the following five steps:

(1) Construction of the class algebra table

(2) Construction of the regular representations of class elements

(3) Finding the eigenvalues and eigenvectors of the representation in step (2)

(4) Arrangement of the eigenvalues into collections $\left\{\lambda^{(\alpha)}\right\}$, corresponding to the given IR's, $\mathscr{D}^{(\alpha)}$

(5) Finding the "columns" in the CT using

$$\chi_j^{(\alpha)} = \frac{l^{(\alpha)}}{{}^{o}C_j} \lambda_j^{(\alpha)}$$

where

$\chi_j^{(\alpha)}$ character of j^{th} class in α^{th} irreducible representation (IR)

$l^{(\alpha)}$ dimension of α^{th} IR

${}^{o}C_j$ order of j^{th} class

A detailed description of each of these steps follows.

Construction of Class Algebra Table

This step in the procedure is conveniently divided into four substeps:

(1) The group $\mathscr{G}$ is broken up into its classes, K_i.

(2) A table is constructed whose columns represent the classes of $\mathscr{G}$ and whose rows are collections of elements containing the inverses of elements of the classes. The collection of inverses of the elements in class K_i is also a class of $\mathscr{G}$ and will be denoted by $K_{\textcircled{i}}$.

(3) The group table is used to find which collections of elements occur when all elements in the class of the i^{th} row $\left(K_{\textcircled{i}}\right)$ operate on all the elements in the class of the j^{th}

column (K_j). (This is the usual convention in group multiplication tables.)

(4) The resulting collection is divided into classes, again being sure to count each class every time it occurs. For example, $K_{(i)}K_j = 2K_0 + 4K_2$ is considered a proper entry in the i^{th} row and J^{th} column.

As an example, the class algebra table of group D_3 is displayed. For D_3, it happens that $K_{(i)} = K_i$. If the group table for D_3 in appendix B is used, the class algebra table for D_3 (also shown in appendix B) is constructed immediately:

	K_0	K_2	K_3
K_0	K_0	K_2	K_3
K_2	K_2	$2K_0 + K_2$	$2K_3$
K_3	K_3	$2K_3$	$3K_0 + 3K_2$

Construction of Regular Representation of Class Elements

Here, use is made of the structure constants of the class algebra $C^j_{i\alpha}$. These are defined by

$$K_iK_\alpha = \sum_{j=1}^{n_c} C^j_{i\alpha}K_j \tag{1}$$

where n_c is the number of classes in $\mathcal{G}$. The regular representation matrix $R(K_\alpha)$ is obtained from the definition

$$R_{ij}(K_\alpha) = C^j_{i\alpha} \tag{2}$$

The class algebra table permits the $C^j_{i\alpha}$ to be "read off" at a glance. The procedure is as follows. The dimension of the regular representation $R(K_\alpha)$ is $n_c \times n_c$. The rows and columns are labeled by the classes of $\mathcal{G}$. Thus, the first row corresponds to the class K_0, the second one to K_2, etc.

The entire representation matrix of the class of K_α is obtained from one column (the α^{th}) of the class algebra table. The second subscript of C identifies the column of the class algebra table which is being considered. The entries in the i^{th} row of $R_{ij}(K_\alpha)$

are the coefficients of the classes in the i^{th} row of the α^{th} column of the class algebra table. These coefficients are equal to the number of times that class appears in the product K_iK_α. Thus, an entry in the α^{th} column of the form $2K_0 + 4K_3$ means that one really has

$$K_iK_\alpha = 2K_0 + 0K_2 + 4K_3 + \ldots$$

so that the i^{th} row of $R_{ij}(K_\alpha)$ is $(2 \quad 0 \quad 4 \ldots)$. Note that the order of the classes in the sequence $K_0, K_2, K_3 \ldots$ must be preserved to obtain the correct representation.

Again the described procedure is illustrated by using the group D_3. There are three classes so there will be three IR's. Although the representation for K_0 is known, it can be used as a check on the structure constants. From the class algebra table in the preceding section

$$K_iK_0 = K_i$$

Since $K_iK_0 = \sum_{j=1}^{3} C^j_{i0}K_j$, it is clear that $C^j_{i0} = \delta_{ij}$. Since $R_{ij}(K_0) = C^j_{i0}$,

$$R(K_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Next $R(K_2)$ will be done in some detail. It is only necessary to examine the K_2 column in the class algebra table and to write each row as a linear combination of classes in the strict sequence K_0, K_2, K_3. The result of this procedure takes the following form:

$$K_0K_2 = \quad K_2 \quad = 0K_0 + 1K_2 + 0K_3$$

$$K_2K_2 = 2K_0 + K_2 = 2K_0 + 1K_2 + 0K_3$$

$$K_3K_2 = \quad 2K_3 \quad = 0K_0 + 0K_2 + 2K_3$$

In this form, the nine structure constants $C^j_{i\alpha}$ are explicitly displayed and

$$R(K_2) = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

In the same way, an examination of the K_3 column of the class algebra table shows that

$$R(K_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 3 & 3 & 0 \end{pmatrix}$$

Eigenvalues of Regular Representation

The standard procedure for finding the eigenvalues of matrices can certainly be used here. However, the collection of matrices which form a representation of a class algebra have special properties. If proper advantage is taken of these properties, the eigenvalues and eigenvectors can be found with far less effort than by using conventional methods.

Conventionally, for groups having a large number of classes, the evaluation of a correspondingly high-order determinant is required to obtain the characteristic equation. A computational technique will be described which may result in a considerable reduction of computation in such cases. This technique is well known but does not appear to be widely used for this purpose.

The method is based on the fact that the first n_c powers of each class K_i of a finite group $\mathcal{G}$ are linearly dependent. The equation expressing this dependence is like the Hamilton-Cayley equation for the class K_i. This equation is readily obtained from the class algebra table and immediately yields the characteristic equation for $R(K_i)$, the regular representation of K_i. The procedure will be illustrated by finding the eigenvalues of $R(K_2)$ of D_3. The results of repeated multiplication of K_2 by itself are listed:

$$\begin{aligned} K_2^0 &= K_0 \\ K_2^1 &= K_2 \\ K_2^2 &= 2K_0 + K_2 \\ K_2^3 = 2K_2 + K_2^2 = 2K_2 + (2K_0 + K_2) &= 2K_0 + 3K_2 \end{aligned}$$

Therefore,

$$K_2^3 - 3K_2^2 + 4K_0 = 0$$

is the Hamilton-Cayley equation for K_2. Consequently,

$$\lambda^3 - 3\lambda^2 + 4 = 0$$

is the characteristic equation for $R(K_2)$. This equation is obtainable directly from $R(K_2)$ by using conventional methods with slightly more algebra. From this result, the eigenvalues $\lambda = 2$, 2, and -1 are obtained.

It is worth noting that the characteristic equation so obtained may not be unique. For example, since $K_2^3 - 3K_2 - 2K_o$ is also equal to zero, another characteristic equation is $\lambda^3 - 3\lambda - 2 = 0$. The roots of this equation are $\lambda = 2$, -1, and -1. The characteristic equation can be relied on to contain all of the distinct eigenvalues (2 and -1 for $R(K_2)$), but the degeneracy may fall on the wrong eigenvalue. This fact is not a serious drawback to the use of this method. In the first place, if some eigenvalues are degenerate, then a characteristic equation yielding only the distinct ones can always be found from a linear relation involving powers of K less than n_c. In the case of $R(K_2)$, the relation $K_2^2 - K_2 - 2K_o = 0$ is valid. Thus, a characteristic equation $\lambda^2 - \lambda - 2 = 0$ may be used to obtain the distinct eigenvalues $\lambda = 2$ and -1. The fact that $\lambda = 2$ is doubly degenerate is important primarily in that two linearly independent eigenvectors belong to the same eigenvalue. This will be shown to emerge automatically in the computation of the eigenvector generators discussed in the following subsection. The main point to be made here is that a Hamilton-Cayley equation may be used to obtain eigenvalues for $R(K_i)$ as soon as a relation involving powers of K_i emerges. If all of the powers up to and including the dimension of $R(K_i)$ are used, all the eigenvalues will be obtained from the resulting characteristic equation. If, while building up powers of K_i, a linear dependence is noticed before $K_i^{n_c}$ is reached, distinct eigenvalues may still be obtained from the algebraically simpler characteristic equation.

This same technique will be used for $R(K_3)$. It will be seen that no relation involving powers of K_3 is obtained until K_3^3 is used:

$$\begin{aligned}
K_3^0 &= K_0 \\
K_3^1 &= K_3 \\
K_3^2 &= 3K_0 + 3K_2 \\
K_3^3 = 3(K_3 + 1K_3K_2) = 3(K_3 + 2K_3) &= 9K_3
\end{aligned}$$

From this list of powers of K_3, the Hamilton-Cayley equation $K_3^3 - 9K_3 = 0$ results. The characteristic equation $\lambda^3 - 9\lambda = 0$ yields the eigenvalues $\lambda = 0$ and ± 3.

Eigenvectors of Regular Representation

As in the preceding section, a procedure will be described which, while not new, does not seem to be widely used for the purpose at hand and does seem to be a rather efficient way to find the eigenvectors. One simply constructs what, in this report, will be called the eigenvector generators $G^{(i)}_{\lambda_j}$, which are defined by

$$G^{(i)}_{\lambda_j} = \prod_{\lambda_k \neq \lambda_j} \left[R(K_i) - \lambda_k I\right]$$

where I is the unit matrix. The matrix $G^{(i)}_{\lambda_j}$ contains, as columns, all of the eigenvectors of $R(K_i)$ belonging to the eigenvalue λ_j. This quantity is directly proportional to Harter's unit dyads (ref. 4). The number of linearly independent columns (or eigenvectors) is equal to the degeneracy of λ_j. This is the reason that there is no loss of information about the degeneracy of the eigenvalues in using the Hamilton-Cayley equation $K_2^2 - K_2 - 2K_0$ to find eigenvalues of $R(K_2)$. The fact that $\lambda = 2$ is doubly degenerate immediately shows up in the form of $G^{(2)}_2$. Thus,

$$G^{(2)}_2 = \left[R(K_2) - (-1)I\right] = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

It is clear that $G^{(2)}_2$ has two linearly independent columns, so $\lambda = 2$ is doubly degenerate. It is convenient to define the symbol $V^{(i)}_{\lambda_j,k}$ as the k^{th} linearly independent column of $G^{(i)}_{\lambda_j}$. These quantities are the eigenvectors. If this procedure for finding eigenvectors is unfamiliar, one may note that

$$V^{(2)}_{2,1} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

and

$$V^{(2)}_{2,2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

are independent column vectors of $G^{(2)}_2$. Also,

$$R(K_2)\begin{pmatrix}1\\2\\0\end{pmatrix}=\begin{pmatrix}2\\4\\0\end{pmatrix}=2V^{(2)}_{2,1}$$

and

$$R(K_2)\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\2\end{pmatrix}=2V^{(2)}_{2,2}$$

Therefore, the correctness of the assertion that $V^{(2)}_{2,1}$ and $V^{(2)}_{2,2}$ are eigenvectors of $R(K_2)$ belonging to the eigenvalue $\lambda = 2$ has been demonstrated. Similarly,

$$G^{(2)}_{-1} = \left[R(K_2) - 2I\right] = \begin{pmatrix}-2 & 1 & 0\\ 2 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}$$

so that only the eigenvector

$$V^{(2)}_{-1} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$$

is obtained this time.

Before computing the $G^{(3)}$'s, it is well to observe that the regular representation of the classes K_i is a faithful one so that

$$R(K_i)R(K_j) = R(K_iK_j) = \sum_{j=1}^{n_c} C^{j}_{i\alpha}R(K_j)$$

Thus, the class algebra table may be used to avoid multiplying matrices in cases where more than two distinct eigenvalues exist for $R(K_i)$. As an example,

$$G^{(3)}_0 = \left[R(K_3) - 3I\right]\left[R(K_3) - (-3I)\right] = R\left(K_3\right)^2 - 9I$$

But,

$$R\left(K_3\right)^2 = R\left(K_3^2\right) = R(3K_0 + 3K_2) = 3R(K_0) + 3R(K_2)$$

Thus,

$$G_0^{(3)} = 3R(K_2) - 6I = 3 \begin{pmatrix} -2 & 1 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

No new eigenvectors are obtained $\left(V_0^{(3)} = V_{-1}^{(2)}\right)$. However, from $G_3^{(3)}$ and $G_{-3}^{(3)}$, the eigenvectors

$$V_3^{(3)} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

and

$$V_{-3}^{(3)} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

are obtained.

Matching the Eigenvalues

A character table (CT) is in effect a collection of traces of IR's of the group. As such, all of the entries in a given row of a CT belong to the same IR. Up to now the eigenvalues are arranged in sets $\left\{\lambda_i\right\}$ according to classes K_i. For a specific IR, $\mathscr{D}^{(\alpha)}$, the character $\chi_i^{(\alpha)}$ assigned to class i is associated with a specific member of the set $\left\{\lambda_i\right\}$. It is therefore required that for a given $\mathscr{D}^{(\alpha)}$, a single eigenvalue be picked from each of the n_c sets $\left\{\lambda_i\right\}$ and that these eigenvalues be arranged in a new set

$$\left\{\lambda^{(\alpha)}\right\} = \lambda_0^{(\alpha)}, \lambda_1^{(\alpha)}, \ldots \lambda_{n_c}^{(\alpha)}$$

all of which will then be associated with the given $\mathscr{D}^{(\alpha)}$. Such a procedure will be called matching the eigenvalues.

As a guide to matching, it may be noted that $V_0^{(3)}$ was equal to $V_{-1}^{(2)}$ in the preceding section. This means that the column vector $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ was an eigenvector of $R(K_2)$

belonging to the eigenvalue $\lambda = -1$ and simultaneously it was an eigenvector of $R(K_3)$ belonging to the eigenvalue $\lambda = 0$. This is not an accidental result. The collection of eigenvalues $\{\lambda^\alpha\}$ has associated with it a single column vector V^α which has the property

$$R(K_i)V^\alpha = \lambda_i^\alpha V^\alpha \qquad i = 0, 2, \ldots, n_c$$

The vector V^α is thus a simultaneous eigenvector of every $R(K_i)$. When this property is used, $\lambda = 0$ from $R(K_3)$ and $\lambda = -1$ from $R(K_2)$ belong to the same set $\{\lambda^\alpha\}$. Every set $\{\lambda^\alpha\}$ will contain the n_c-fold degenerate eigenvalues $\lambda = 1$ from $R(K_0)$ so that the complete set found is

K_0	K_2	K_3
$\lambda_0^\alpha = 1$	$\lambda_2^\alpha = -1$	$\lambda_3^\alpha = 0$

Similarly, one finds that both $V_3^{(3)}$ and $V_{-3}^{(3)}$ are eigenvectors of $R(K_2)$ belonging to the same eigenvalue $\lambda = 2$, so that the remaining matched sets are

K_0	K_2	K_3
1	2	3
1	2	-3

(It turns out that neither $V_{2,1}^{(2)}$ nor $V_{2,2}^{(2)}$ are eigenvectors of $R(K_3)$. However, the linear combinations $V_{2,1}^{(2)} + V_{2,2}^{(2)}\left(= V_2^{(2)}\right)$ and $V_{2,1}^{(2)} - V_{2,2}^{(2)}\left(= V_3^{(3)}\right)$ are simultaneous vectors of $R(K_2)$ and $R(K_3)$.)

The rest is a matter of convention. The most common convention is the listing of the characters in order of increasing dimension of the IR. The relation

$$\frac{1}{{}^o\mathcal{G}} \sum_j \frac{\left[\lambda_j^{(\alpha)}\right]^2}{{}^oC_j} = \frac{1}{\left[l^{(\alpha)}\right]^2}$$

is valuable in arranging the table. If the matched set

$$\begin{array}{cccc} & K_0 & K_2 & K_3 \\ \lambda^{(0)} = & 1 & 2 & 3 \end{array}$$

is picked to be associated with $\mathscr{D}^{(0)}$, then it is found that

$$\frac{1}{\left[l^{(0)}\right]^2} = 1$$

Similarly, if $\left\{\lambda^{(2)}\right\} = 1, 2, -3$ and $\left\{\lambda^{(3)}\right\} = 1, -1, 0$ (where the sequence in each case corresponds to K_0, K_2, K_3) are used, it is found that $l^{(2)} = 1$ and $l^{(3)} = 2$.

Columns of Character Table

All of the necessary numbers are now available. Substitution of these numbers into the relation

$$\chi_j^{(\alpha)} = \frac{l^{(\alpha)}}{{}^{o}C_j} \lambda_j^{(\alpha)}$$

allows construction of the following completed character table:

	K_0	K_2	K_3
$\mathscr{D}^{(0)}$	$\chi_0^{(0)} = 1$	$\chi_2^{(0)} = 1$	$\chi_3^{(0)} = 1$
$\mathscr{D}^{(1)}$	$\chi_0^{(2)} = 1$	$\chi_2^{(2)} = 1$	$\chi_3^{(2)} = -1$
$\mathscr{D}^{(2)}$	$\chi_2^{(3)} = 2$	$\chi_2^{(3)} = 1$	$\chi_3^{(3)} = 0$

Character Table for D_4 (~Quaternion Group Q)

In the preceding description of the procedure used in obtaining character tables, D_3 was used to illustrate each step. After all the steps were completed, the character table of D_3 was displayed. Another example is now worked out in detail - the character table for D_4.

The group table for D_4 is shown in appendix B. From it, the following class algebra table may be readily constructed:

K_0	K_2	K_3	K_4	K_5
K_0	K_2	K_3	K_4	K_5
K_2	K_0	K_3	K_4	K_5
K_3	K_3	$2K_0 + 2K_2$	$2K_5$	$2K_4$
K_4	K_4	$2K_5$	$2K_0 + 2K_2$	$2K_3$
K_5	K_5	$2K_4$	$2K_3$	$2K_0 + 2K_2$

Since there are five classes, the regular representation of the classes consists of 5 by 5 matrices. These may be constructed from the structure constants displayed in this class algebra table. They are as follows:

$$R(K_0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad R(K_2) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad R(K_3) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

$$R(K_4) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix} \qquad R(K_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \end{pmatrix}$$

The eigenvalues of $R(K_2)$ can be obtained easily directly from the matrix itself. Thus,

$$(\lambda - 1)^3(\lambda^2 - 1) = 0$$

is the characteristic equation and $\lambda = 1, 1, 1, 1, -1$ are the eigenvalues:

$$G_1^{(2)} = \left[R(K_2) - (-1)I\right] = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Therefore, as expected, there are four linearly independent eigenvectors belonging to $\lambda = 1$:

$$V^{(2)}_{1,1} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad V^{(2)}_{1,2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad V^{(2)}_{1,3} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad V^{(2)}_{1,4} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The eigenvector generator for $\lambda = -1$ is

$$G_{-1}^{(2)} = \left[R(K_3) - I\right] = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore,

$$V_{-1}^{(2)} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The eigenvalue $\lambda = -1$ is nondegenerate. Therefore, we may try operating on it with $R(K_3)$ to see if it is an eigenvector of $R(K_3)$ also:

$$R(K_3)V_{-1}^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0V_{-1}^{(2)}$$

Therefore $V_{-1}^{(2)}$ is an eigenvector of $R(K_3)$ belonging to the eigenvalue $\lambda = 0$ of $R(K_3)$. Thus, the matching

K_2	K_3
-1	0

has resulted.

Operating on $V_{-1}^{(2)}$ by $R(K_4)$ and $R(K_5)$ would show that $V_{-1}^{(2)}$ is also an eigenvector belonging $\lambda = 0$ of each of these matrices also so that a complete matching set $\left\{\lambda^{(\alpha)}\right\}$ is obtained:

K_0	K_2	K_3	K_4	K_5
1	-1	0	0	0

In this example, however, repetitious computation may be avoided by noting that the class structures of K_3, K_4, and K_5 are all the same; that is,

$$K_3^2 = K_4^2 = K_5^2 = 2K_0 + 2K_2$$

Thus, the same Hamilton-Cayley equation (and therefore, the same characteristic equation) is obtained for each of these classes. However, the eigenvector generators are not the same.

The characteristic equation for each of the classes is

$$\lambda^3 - 4\lambda = 0$$

Therefore, $\lambda = 0, \pm 2$ are the distinct eigenvalues.

In order to find the degeneracy of these eigenvalues, the eigenvector generator for K_3 is examined:

$$G_0^{(3)} = \left[R(K_3) - 2I\right]\left[R(K_3) - (-2)I\right] = 2\left[R(K_2) - R(K_0)\right]$$

(Note that since $R\left(K_4\right)^2 = R\left(K_5\right)^2 = 2R(K_0) + 2R(K_2)$, $G_0^{(4)}$ and $G_0^{(5)}$ are also the same as $G_0^{(3)}$.)

$$G_0^{(3)} = 2\begin{pmatrix} 1 & -1 & \cdot & & \cdot \\ -1 & 1 & \cdot & 0 & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 & \cdot \\ 0 & 0 & \cdot & & \cdot \end{pmatrix}$$

Therefore, there is only one eigenvector and $\lambda = 0$ is nondegenerate. $\left(V_0^{(3)} = V_{-1}^{(2)}\right.$, as was found earlier.$\left.\right)$

Next $G_2^{(3)}$ is examined:

$$G_2^{(3)} = \left[R(K_3) - 0 \cdot I\right]\left[R(K_3) + 2I\right] = 2\left[R(K_0) + R(K_2) + R(K_3)\right]$$

$$= 2\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 \end{pmatrix}$$

Thus, $\lambda = 2$ is doubly degenerate:

$$V_{2,1}^{(3)} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \qquad V_{2,2}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Now for D_4, each $R(K_i)$ has five roots in its characteristic equation. For $R(K_3)$, $\lambda = 0$ is nondegenerate and $\lambda = 2$ is doubly degenerate, so the remaining two roots must both belong to $\lambda = -2$. Therefore, $\lambda = -2$ is doubly degenerate. However, $G_{-2}^{(3)}$ must be examined in order to obtain $V_{-2,1}^{(3)}$ and $V_{-2,2}^{(3)}$ explicitly:

$$G_{-2}^{(3)} = \begin{pmatrix} 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -2 & 2 \end{pmatrix}$$

so that

$$V^{(3)}_{-2,1} = \begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \\ 0 \end{pmatrix} \qquad V^{(3)}_{-2,2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

The eigenvector generators for $R(K_4)$ and $R(K_5)$ are different from those of $R(K_3)$. Thus,

$$G^{(4)}_2 = 2\left[R(K_0) + R(K_2) + R(K_4)\right] = 2\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 2 \\ 2 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 2 \end{pmatrix}$$

$$G^{(4)}_{-2} = 2\left[R(K_0) + R(K_2) - R(K_4)\right] = 2\begin{pmatrix} 1 & 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & -2 \\ -2 & -2 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 & 2 \end{pmatrix}$$

and

$$V^{(4)}_{2,1} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} \qquad V^{(4)}_{2,2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \qquad V^{(4)}_{-2,1} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \\ 0 \end{pmatrix} \qquad V^{(4)}_{-2,2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$$G^{(5)}_2 = 2\left[R(K_0) + R(K_2) + R(K_5)\right] = 2\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 2 & 2 & 0 & 0 & 2 \end{pmatrix}$$

$$G^{(5)}_{-2} = 2\left[R(K_0) + R(K_2) - R(K_5)\right] = 2\begin{pmatrix} 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & -2 & 2 & 0 \\ -2 & -2 & 0 & 0 & 2 \end{pmatrix}$$

and

$$V^{(5)}_{2,1} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} \qquad V^{(5)}_{2,2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \qquad V^{(5)}_{-2,1} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \qquad V^{(5)}_{-2,2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

The rest of the matching in this case is not entirely a mechanical procedure. Each simultaneous eigenvector which can be used to obtain a set $\{\lambda^{(\alpha)}\}$ is expressible as a particular linear combination of eigenvectors of a given $R(K_i)$ belonging to the degenerate eigenvalue λ_i of that matrix $R(K_i)$. A certain amount of trial and error is required in such cases to find the simultaneous eigenvectors. However, some general observations can reduce the total number of trials (and thereby some of the "error"). For instance, each of the eigenvectors of $R(K_3)$, $R(K_4)$, and $R(K_5)$ belonging to degenerate eigenvalues are expressible as linear combinations of the four eigenvectors of $R(K_2)$ belonging to $\lambda = 1$. Therefore, any linear combination of such $V^{(3)}$'s, $V^{(4)}$'s, and $V^{(5)}$'s is automatically an eigenvector of $R(K_2)$ belonging to $\lambda = 1$.

These comments may be readily applied to the construction of the simultaneous eigenvector

$$V^S_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 2 \end{pmatrix}$$

The linear combinations of eigenvectors of $R(K_3)$, $R(K_4)$, and $R(K_5)$ which go to make up this vector are

$$V^S_1 = V^{(3)}_{2,1} + 2V^{(3)}_{2,2} = V^{(4)}_{2,1} + 2V^{(4)}_{2,2} = V^{(5)}_{2,1} + 2V^{(5)}_{2,2} = V^{(2)}_{1,1} + 2V^{(2)}_{1,2} + 2V^{(1)}_{1,3} + 2V^{(1)}_{1,4}$$

Therefore, another matched set is obtained. Namely,

K_2	K_3	K_4	K_5
1	2	2	2

Similarly, trying

$$V^{(3)}_{2,1} - 2V^{(3)}_{2,2} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ -2 \\ -2 \end{pmatrix}$$

shows that

$$V^S_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ -2 \\ -2 \end{pmatrix} = V^{(4)}_{-2,1} + 2V^{(4)}_{-2,2} = V^{(5)}_{-2,1} + 2V^{(5)}_{-2,2}$$

$$= V^{(2)}_{1,1} + 2V^{(2)}_{1,2} - 2V^{(2)}_{1,3} - 2V^{(2)}_{1,4}$$

so that another matched set is thereby obtained:

K_2	K_3	K_4	K_5
1	2	-2	-2

The trials $V^{(3)}_{-2,1} + 2V^{(3)}_{-2,2}$ and $V^{(3)}_{-2,2} - 2V^{(3)}_{-2,2}$, respectively, are found to result in the remaining two linearly independent simultaneous eigenvectors

$$V^S_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \\ 2 \\ -2 \end{pmatrix} \qquad V^S_4 = \begin{pmatrix} 1 \\ 1 \\ -2 \\ -2 \\ 2 \end{pmatrix}$$

which yield the remaining matched sets

K_2	K_3	K_4	K_5
1	-2	2	-2
1	-2	-2	2

If the nondegenerate eigenvector $V_{-1}^{(2)}$ is called V_5^S, these matched sets can now be used to write the diagonalized $R(K_i)$ matrices in a form suitable for obtaining the characters of D_4. These are

$$R(K_1) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 0 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 0 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} \quad R(K_2) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & 0 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 0 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 \end{pmatrix} \quad R(K_3) = \begin{pmatrix} 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & 0 & \cdot \\ \cdot & \cdot & -2 & \cdot & \cdot \\ \cdot & 0 & \cdot & -2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

$$R(K_4) = \begin{pmatrix} 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & -2 & \cdot & 0 & \cdot \\ \cdot & \cdot & 2 & \cdot & \cdot \\ \cdot & 0 & \cdot & -2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix} \quad R(K_5) = \begin{pmatrix} 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & -2 & \cdot & 0 & \cdot \\ \cdot & \cdot & -2 & \cdot & \cdot \\ \cdot & 0 & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

There are five classes and therefore five IR's for D_4. Therefore, the only solution to the equation $\sum_{\alpha=1}^{5} \left[l^{(\alpha)} \right]^2 = {}^{o}\mathcal{G} = 8$ is $l^{(0)} = l^{(2)} = l^{(3)} = l^{(4)} = 1$ and $l^{(5)} = 2$. The class structure is such that ${}^{o}C_0 = {}^{o}C_2 = 1$ and ${}^{o}C_3 = {}^{o}C_4 = {}^{o}C_5 = 2$. The character table for D_4 is now obtained immediately (note that $\mathscr{D}^{(i)_j}$ denotes j^{th} IR of dimension $i \times i$, not counting $\mathscr{D}^{(0)}$ as a 1×1):

	K_0	K_2	K_3	K_4	K_5
$\mathscr{D}^{(0)}$	$\chi_0^{(0)} = 1$	$\chi_2^{(0)} = 1$	$\chi_3^{(0)} = 1$	$\chi_4^{(0)} = 1$	$\chi_5^{(0)} = 1$
$\mathscr{D}^{(1_1)}$	$\chi_0^{(2)} = 1$	$\chi_2^{(2)} = 1$	$\chi_3^{(2)} = 1$	$\chi_4^{(2)} = -1$	$\chi_5^{(2)} = -1$
$\mathscr{D}^{(1_2)}$	$\chi_0^{(3)} = 1$	$\chi_2^{(3)} = 1$	$\chi_3^{(3)} = -1$	$\chi_4^{(3)} = 1$	$\chi_5^{(3)} = -1$
$\mathscr{D}^{(1_3)}$	$\chi_0^{(4)} = 1$	$\chi_2^{(4)} = 1$	$\chi_3^{(4)} = -1$	$\chi_4^{(4)} = -1$	$\chi_5^{(4)} = 1$
$\mathscr{D}^{(2)}$	$\chi_0^{(5)} = 2$	$\chi_2^{(5)} = -2$	$\chi_3^{(5)} = 0$	$\chi_4^{(5)} = 0$	$\chi_5^{(5)} = 0$

CONCLUDING REMARKS

A systematic procedure for constructing the character table of a given finite group is presented. Although the individual sections of the procedure are not original, the collection of procedures would seem to be justified on the grounds that they make the task of computing group character tables much more straightforward than previously published procedures. Each step in the construction of character tables is illustrated by worked out examples. An attempt was made to make the report self-contained by including an appendix of group tables, character tables, and class algebra tables for many of the common finite groups.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, May 22, 1968,
124-09-01-05-22.

APPENDIX A

SYMBOLS

CT	character table
$C^j_{i\alpha}$	structure constant for an algebra
${}^{(o)}C_j$	number of elements in j^{th} class (order of class)
$\mathscr{D}^{(\alpha)}$	α^{th} irreducible representation of a group $\mathscr{G}$
$G^i_{\lambda_j}$	eigenvector generator for class K_i which generated eigenvectors belonging to eigenvalue λ_j of representation $R(K_i)$
$\mathscr{G}$	group
g	element of group
IR	irreducible representation
K_i	i^{th} class of group
$l^{(\alpha)}$	linear dimension of $\mathscr{D}^{(\alpha)}$
n_c	number of classes
$R(K_i)$	regular representation of the class algebra for class K_i
$\mathscr{R}$	representation
Tr	trace or sum of diagonal elements of a matrix
$V^i_{\lambda_j,k}$	k^{th} eigenvector belonging to eigenvalue λ_j of representation $R(K_i)$
λ	eigenvalue
χ	character; trace of irreducible representation
ω	root of 1 (For a cyclic group of order n, $\omega = e^{2\pi i/n}$.)
$\sim$	is isomorphic to (e.g., $D_4 \sim Q$)

Left superscript:

o	order of group

Right superscripts:

i_j	on irreducible representations the j^{th} irreducible representations of dimension $i \times i$, not counting $D^{(0)}$ as a 1×1
0	on irreducible representations the irreducible representation which represents every element of the group by 1
S	simultaneous eigenvector

Right subscript:

(i)	denotes fact that attached symbol is for the inverse of that for subscript i

Groups:

C_n	cyclic group of order n
D_n	n^{th} dihedral group
T	tetrahedral group
T_d	cube group

Group elements:

R, r	rotations
ρ	reflections
i	inversions (reflection through origin)

APPENDIX B

GROUP TABLES, CHARACTER TABLES, AND CLASS ALGEBRA TABLES OF SOME COMMON FINITE GROUPS

The material in this appendix is taken from lectures delivered by W. G. Harter at NASA in the summer of 1966. As is usual, an entry in a table is the result of group operation by the element in the column heading the entry followed by group operation by the element in the row heading the entry.

CYCLIC GROUPS, C_n

(Each element of C_n is in a class by itself, $\omega = \exp(2\pi i/n)$)

Group Table / Character Table

C_2

	1	R
1	1	R
R	R	1

	K_0	K_2
	1	R
$\mathscr{D}^{(0)}$	1	1
$\mathscr{D}^{(1)}$	1	-1

C_3

	1	R	R^2
1	1	R	R^2
R^2	R^2	1	R
R	R	R^2	1

	K_0	K_2	K_3
	1	R	R^2
$\mathscr{D}^{(0)}$	1	1	1
$\mathscr{D}^{(\omega_1)}$	1	ω	ω^2
$\mathscr{D}^{(\omega_2)}$	1	ω^2	ω

C_4

	1	R	R^2	R^3
1	1	R	R^2	R^3
R^3	R^3	1	R	R^2
R^2	R^2	R^3	1	R
R	R	R^2	R^3	1

	K_0	K_2	K_3	K_4
	1	R	R^2	R^3
$\mathscr{D}^{(0)}$	1	1	1	1
$\mathscr{D}^{(\omega_1)}$	1	i	-1	-i
$\mathscr{D}^{(\omega_2)}$	1	-1	1	-1
$\mathscr{D}^{(\omega_3)}$	1	-i	-1	i

GROUP D_2

Group Table

$(g_{ⓘ} = g_i,\ i = 1, \ldots, 4)$

	1	R_1^2	R_2^2	R_3^2
1	1	R_1^2	R_2^2	R_3^2
R_1^2	R_1^2	1	R_3^2	R_2^2
R_2^2	R_2^2	R_3^2	1	R_1^2
R_3	R_3^2	R_2^2	R_1^2	1

Character Table

	K_0	K_2	K_3	K_4
	1	R_1^2	R_2^2	R_4^2
$\mathscr{D}^{(0)}$	1	1	1	1
$\mathscr{D}^{(1_1)}$	1	1	-1	-1
$\mathscr{D}^{(1_2)}$	1	-1	1	-1
$\mathscr{D}^{(1_3)}$	1	-1	-1	1

GROUP D_3

Group Table

	g_0	g_2	g_3	g_4	g_5	g_6
	1	R	R^2	ρ_1	ρ_2	ρ_3
$g_{⓪} = 1$	1	R	R^2	ρ_1	ρ_2	ρ_3
$g_{②} = R^2$	R^2	1	R	ρ_3	ρ_1	ρ_2
$g_{③} = R$	R	R^2	1	ρ_2	ρ_3	ρ_1
$g_{④} = \rho_1$	ρ_1	ρ_3	ρ_2	1	R^2	R
$g_{⑤} = \rho_2$	ρ_2	ρ_1	ρ_3	R	1	R^2
$g_{⑥} = \rho_3$	ρ_3	ρ_2	ρ_1	R^2	R	1

Character Table

	K_0	K_2	K_3
		$\{R, R^2\}$	$\{\rho_i\}$
$\mathscr{D}^{(0)}$	1	1	1
$\mathscr{D}^{(1)}$	1	1	-1
$\mathscr{D}^{(2)}$	2	-1	0

Class Algebra Table

K_0	K_2	K_3
K_2	$2K_0 + K_2$	$2K_3$
K_3	$2K_3$	$3K_0 + 3K_2$

GROUPS D_4 AND Q

Group Table for D_4

	g_0	g_2	g_3	g_4	g_5	g_6	g_7	g_8
	1	R^2	R	R^3	ρ_1	ρ_2	ρ_3	ρ_4
g_0	1	R^2	R	R^3	ρ_1	ρ_2	ρ_3	ρ_4
g_2	R^2	1	R^3	R	ρ_2	ρ_1	ρ_4	ρ_3
g_3	R^3	R	1	R^2	ρ_3	ρ_4	ρ_2	ρ_1
g_4	R	R^3	R^2	1	ρ_4	ρ_3	ρ_1	ρ_2
g_5	ρ_1	ρ_2	ρ_3	ρ_4	1	R^2	R	R^3
g_6	ρ_2	ρ_1	ρ_4	ρ_3	R^2	1	R^3	R
g_7	ρ_3	ρ_4	ρ_2	ρ_1	R^3	R	1	R^2
g_8	ρ_4	ρ_3	ρ_1	ρ_2	R	R^3	R^2	1

Group Table for Q

	g_0	g_2	g_3	g_4	g_5	g_6	g_7	g_8
$g_0 = 1$	1	-1	i	-i	j	-j	k	-k
$g_2 = -1$	-1	1	-i	i	-j	j	-k	k
$g_3 = -i$	-i	i	1	-1	-k	k	j	-j
$g_4 = i$	i	-i	-1	1	k	-k	-j	j
$g_5 = -j$	-j	j	k	-k	1	-1	-i	i
$g_6 = j$	j	-j	-k	k	-1	1	i	-i
$g_7 = -k$	-k	k	-j	j	i	-i	1	-1
$g_8 = k$	k	-k	j	-j	-i	i	-1	1

Character Table for D_4 or Q

	K_0	K_2	K_3	K_4	K_5
$\mathcal{D}^{(0)}$	1	1	1	1	1
$\mathcal{D}^{(1_1)}$	1	1	1	-1	-1
$\mathcal{D}^{(1_2)}$	1	1	-1	1	-1
$\mathcal{D}^{(1_3)}$	1	1	-1	-1	1
$\mathcal{D}^{(2)}$	2	-2	0	0	0

Class Algebra Table for D_4 or Q

	K_0	K_2	K_3	K_4	K_5
D_4	1	R^2	$\{R, R^3\}$	$\{\rho_1, \rho_2\}$	$\{\rho_3, \rho_4\}$
Q	1	-1	$\{i, -i\}$	$\{j, -j\}$	$\{k, -k\}$
	K_0	K_2	K_3	K_4	K_5
	K_2	K_0	K_3	K_4	K_5
	K_3	K_3	$2K_0 + 2K_2$	$2K_5$	$2K_4$
	K_4	K_4	$2K_5$	$2K_0 + 2K_2$	$2K_3$
	K_5	K_5	$2K_4$	$2K_3$	$2K_0 + 2K_2$

GROUP D_5

Group Table

	g_0	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}
	1	R	R^4	R^2	R^3	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
g_{0}	1	R	R^4	R^2	R^3	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5
g_{2}	R^4	1	R^3	R	R^2	ρ_5	ρ_1	ρ_2	ρ_3	ρ_4
g_{3}	R	R^2	1	R^3	R^4	ρ_2	ρ_3	ρ_4	ρ_5	ρ_1
g_{4}	R^3	R^4	R^2	1	R	ρ_4	ρ_5	ρ_1	ρ_2	ρ_3
g_{5}	R^2	R^3	R	R^4	1	ρ_3	ρ_4	ρ_5	ρ_1	ρ_2
g_{6}	ρ_1	ρ_5	ρ_2	ρ_4	ρ_3	1	R^4	R^3	R^2	R
g_{7}	ρ_2	ρ_1	ρ_3	ρ_5	ρ_4	R	1	R^4	R^3	R^2
g_{8}	ρ_3	ρ_2	ρ_4	ρ_1	ρ_5	R^2	R	1	R^4	R^3
g_{9}	ρ_4	ρ_3	ρ_5	ρ_2	ρ_1	R^3	R^2	R	1	R^4
g_{10}	ρ_5	ρ_4	ρ_1	ρ_3	ρ_2	R^4	R^3	R^2	R	1

Character Table

	K_0	K_2	R_3	K_4
$\mathscr{D}^{(0)}$	1	1	1	1
$\mathscr{D}^{(1)}$	1	1	1	-1
$\mathscr{D}^{(2_1)}$	2	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	0
$\mathscr{D}^{(2_2)}$	2	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$	0

Class Algebra Table

	K_0	K_2	K_3	K_4
	1	$\{R, R^4\}$	$\{R^2, R^3\}$	$\{\rho_i\}$
K_0	K_0	K_2	K_3	K_4
K_2	K_2	$2K_0 + K_3$	$K_2 + K_3$	$2K_4$
K_3	K_3	$K_2 + K_3$	$2K_0 + K_2$	$2K_4$
K_4	K_4	$2K_4$	$2K_4$	$5K_0 + 5K_2 + 5K_3$

GROUP T

Group Table

	g_0	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}
	1	r_1	r_2	r_3	r_4	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2
g_{0}	1	r_1	r_2	r_3	r_4	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2
g_{2}	r_1^2	1	R_1^2	R_2^2	R_3^2	r_1	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2
g_{3}	r_2^2	R_1^2	1	R_3^2	R_2^2	r_4	r_2	r_1	r_3	r_3^2	r_1^2	r_4^2
g_{4}	r_3^2	R_2^2	R_3^2	1	R_1^2	r_2	r_4	r_3	r_1	r_2^2	r_4^2	r_1^2
g_{5}	r_4^2	R_3^2	R_2^2	R_1^2	1	r_3	r_1	r_2	r_4	r_1^2	r_3^2	r_2^2
g_{6}	r_1	r_1^2	r_4^2	r_2^2	r_3^2	1	R_2^2	R_3^2	R_1^2	r_2	r_3	r_4
g_{7}	r_2	r_3^2	r_2^2	r_4^2	r_1^2	R_2^2	1	R_1^2	R_3^2	r_1	r_4	r_3
g_{8}	r_3	r_4^2	r_1^2	r_3^2	r_2^2	R_3^2	R_1^2	1	R_2^2	r_4	r_1	r_2
g_{9}	r_4	r_2^2	r_3^2	r_1^2	r_4^2	R_1^2	R_3^2	R_2^2	1	r_3	r_2	r_1
g_{10}	R_1^2	r_4	r_3	r_2	r_1	r_2^2	r_1^2	r_4^2	r_3^2	1	R_3^2	R_2^2
g_{11}	R_2^2	r_2	r_1	r_4	r_3	r_3^2	r_4^2	r_1^2	r_2^2	R_3^2	1	R_1^2
g_{12}	R_3^2	r_3	r_4	r_1	r_2	r_4^2	r_3^2	r_2^2	r_1^2	R_2^2	R_1^2	1

Character Table

	K_0	K_2	K_3	K_4
	1	$\{r_i\}$	$\{r_i^2\}$	$\{R_i^2\}$
$\mathscr{D}^{(0)}$	1	1	1	1
$\mathscr{D}^{(1_1)}$	1	ω	ω^*	1
$\mathscr{D}^{(1_2)}$	1	ω^*	ω	1
$\mathscr{D}^{(2)}$	3	0	0	-1

$\omega = e^{2\pi i/3}$

Class Algebra Table

	K_0	K_2	K_3	K_4
K_{2} =	K_3	$4K_0 + 4K_4$	$4K_2$	$3K_3$
K_{3} =	K_2	$4K_3$	$4K_0 + 4K_4$	$3K_2$
K_{4} =	K_4	$3K_2$	$3K_3$	$3K_0 + 2K_4$

GROUP T_d OR O

Group Table

	g_0	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}	g_{13}	g_{14}	g_{15}	g_{16}	g_{17}	g_{18}	g_{19}	g_{20}	g_{21}	g_{22}	g_{23}	g_{24}
	1	r_1	r_2	r_3	r_4	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2	R_1	R_2	R_3	R_1^3	R_2^3	R_3^3	i_1	i_2	i_3	i_4	i_5	i_6
g_{0}	1	r_1	r_2	r_3	r_4	r_1^2	r_2^2	r_3^2	r_4^2	R_1^2	R_2^2	R_3^2	R_1	R_2	R_3	R_1^3	R_2^3	R_3^3	i_1	i_2	i_3	i_4	i_5	i_6
g_{2}	r_1^2	1	R_1^2	R_2^2	R_3^2	r_1	r_3	r_4	r_2	r_4^2	r_2^2	r_3^2	R_2^3	R_3^3	R_1^3	i_1	i_3	i_6	R_3	i_4	R_1	i_5	i_2	R_2
g_{3}	r_2^2	R_1^2	1	R_3^2	R_2^2	r_4	r_2	r_1	r_3	r_3^2	r_1^2	r_4^2	i_2	i_3	R_1	R_2	R_3^3	i_5	i_4	R_3	R_1^3	i_6	R_2^3	i_1
g_{4}	r_3^2	R_2^2	R_3^2	1	R_1^2	r_2	r_4	r_3	r_1	r_2^2	r_4^2	r_1^2	R_2	i_4	i_6	i_2	R_3	R_1^3	i_3	R_3^3	i_5	R_1	i_1	R_2^3
g_{5}	r_4^2	R_3^2	R_2^2	R_1^2	1	r_3	r_1	r_2	r_4	r_1^2	r_3^2	r_2^2	i_1	R_3	i_5	R_2^3	i_4	R_1	R_3^3	i_3	i_6	R_1^3	R_2	i_2
g_{6}	r_1	r_1^2	r_4^2	r_2^2	r_3^2	1	R_2^2	R_3^2	R_1^2	r_2	r_3	r_4	i_3	i_6	i_1	R_3	R_1	R_2	R_1^3	i_5	R_2^3	i_2	i_4	R_3^3
g_{7}	r_2	r_3^2	r_2^2	r_4^2	r_1^2	R_2^2	1	R_1^2	R_3^2	r_1	r_4	r_3	R_3	R_1^3	i_2	i_3	i_5	R_2^3	i_6	R_1	R_2	i_1	R_3^3	i_4
g_{8}	r_3	r_4^2	r_1^2	r_3^2	r_2^2	R_3^2	R_1^2	1	R_2^2	r_4	r_1	r_2	i_4	R_1	R_2^3	R_3^3	i_6	i_2	i_5	R_1^3	i_1	R_2	i_3	R_3
g_{9}	r_4	r_2^2	r_3^2	r_1^2	r_4^2	R_1^2	R_3^2	R_2^2	1	r_3	r_2	r_1	R_3^3	i_5	R_2	i_4	R_1^3	i_1	R_1	i_6	i_2	R_2^3	R_3	i_3
g_{10}	R_1^2	r_4	r_3	r_2	r_1	r_2^2	r_1^2	r_4^2	r_3^2	1	R_3^2	R_2^2	R_1^3	i_1	i_4	R_1	i_2	i_3	R_2	R_2^3	R_3^3	R_3	i_6	i_5
g_{11}	R_2^2	r_2	r_1	r_4	r_3	r_3^2	r_4^2	r_1^2	r_2^2	R_3^2	1	R_1^2	i_5	R_2^3	i_3	i_6	R_2	i_4	i_2	i_i	R_3	R_3^3	R_1	R_1^3
g_{12}	R_3^2	r_3	r_4	r_1	r_2	r_4^2	r_3^2	r_2^2	r_1^2	R_2^2	R_1^2	1	i_6	i_2	R_3^3	i_5	i_1	R_3	R_2^3	R_2	i_4	i_3	R_1^3	R_1
g_{13}	R_1^3	R_2	i_2	R_2^3	i_1	i_3	R_3^3	i_4	R_3	R_1	i_5	i_6	1	r_4	r_3^2	R_1^2	r_2	r_1^2	r_1	r_3	r_2^2	r_4^2	R_3^2	R_2^2
g_{14}	R_2^3	R_3	i_3	i_4	R_3^3	i_6	R_1	R_1^3	i_5	i_1	R_2	i_2	r_4^2	1	r_2	r_1^2	R_2^2	r_3	R_3^2	R_1^2	r_1	r_4	r_2^2	r_3^2
g_{15}	R_3^3	R_1	R_1^3	i_6	i_5	i_1	i_2	R_2	R_2^3	i_4	i_3	R_3	r_3	r_2^2	1	r_4	r_1^2	R_3^2	r_4^2	r_3^2	R_1^2	R_2^2	r_2	r_1
g_{16}	R_1	i_1	R_2^3	i_2	R_2	R_3^3	i_3	R_3	i_4	R_1^3	i_6	i_5	R_1^2	r_1	r_4^2	1	r_3	r_2^2	r_4	r_2	r_1^2	r_3^2	R_2^2	R_3^2
g_{17}	R_2	i_3	R_3	R_3^3	i_4	R_1^3	i_5	i_6	R_1	i_2	R_2^3	i_1	r_2^2	R_2^2	r_1	r_3^2	1	r_4	R_1^2	R_3^2	r_2	r_3	r_4^2	r_1^2
g_{18}	R_3	i_6	i_5	R_1	R_1^3	R_2^3	R_2	i_2	i_1	i_3	i_4	R_3^3	r_1	r_3^2	R_3^2	r_2	r_4^2	1	r_1^2	r_2^2	R_2^2	R_1^2	r_4	r_3
g_{19}	i_1	R_3^3	i_4	i_3	R_3	R_1	i_6	i_5	R_1^3	R_2^3	i_2	R_2	r_1^2	R_3^2	r_4	r_4^2	R_1^2	r_1	1	R_2^2	r_3	r_2	r_3^2	r_2^2
g_{20}	i_2	i_4	R_3^3	R_3	i_3	i_5	R_1^3	R_1	i_6	R_2	i_1	R_2^3	r_3^2	R_1^2	r_3	r_2^2	R_3^2	r_2	R_2^2	1	r_4	r_1	r_1^2	r_4^2
g_{21}	i_3	R_1^3	R_1	i_5	i_6	R_2	R_2^3	i_1	i_2	R_3	R_3^3	i_4	r_2	r_1^2	R_1^2	r_1	r_2^2	R_2^2	r_3^2	r_4^2	1	R_3^2	r_3	r_4
g_{22}	i_4	i_5	i_6	R_1^3	R_1	i_1	i_2	R_2^3	R_2	R_3^3	R_3	i_3	r_4	r_4^2	R_2^2	r_3	r_3^2	R_1^2	r_2^2	r_1^2	R_3^2	1	r_1	r_2
g_{23}	i_5	i_2	R_2	i_1	R_2^3	i_4	R_3	i_3	R_3^3	i_6	R_1^3	R_1	R_3^2	r_2	r_2^2	R_2^2	r_4	r_4^2	r_3	r_1	r_3^2	r_1^2	1	R_1^2
g_{24}	i_6	R_2^3	i_1	R_2	i_2	R_3	i_4	R_3^3	i_3	i_5	R_1	R_1^3	R_2^2	r_3	r_1^2	R_3^2	r_1	r_3^2	r_2	r_4	r_4^2	r_2^2	R_1^2	1

Class Algebra Table

K_0	K_2	K_3	K_4	K_5
K_2	$8K_0 + 4K_2 + 8K_3$	$3K_2$	$4K_4 + 4K_5$	$4K_4 + 4K_5$
K_3	$3K_2$	$3K_0 + 2K_3$	$K_4 + 2K_5$	$2K_4 + K_5$
K_4	$4K_4 + 4K_5$	$K_4 + 2K_5$	$6K_0 + 3K_2 + 2K_3$	$3K_2 + 4K_3$
K_5	$4K_4 + 4K_5$	$2K_4 + K_5$	$3K_2 + 4K_3$	$6K_0 + 3K_2 + 2K_3$

Character Table

	K_0	K_2	K_3	K_4	K_5
	1	$\{r_j, r_j^2\}$	$\{R_j^2\}$	$\{R_j, R_j^3\}$	$\{i_j\}$
$\mathcal{G}^{(0)}$	1	1	1	1	1
$\mathcal{G}^{(1)}$	1	1	1	-1	-1
$\mathcal{G}^{(2)}$	2	-1	2	0	0
$\mathcal{G}^{(3_1)}$	3	0	-1	-1	1
$\mathcal{G}^{(3_2)}$	3	0	-1	1	-1

REFERENCES

1. Margenau, Henry; and Murphy, George M.: The Mathematics of Physics and Chemistry. Second ed., D. Van Nostrand, Inc., 1956, pp. 526-530.

2. Hamermesh, Morton: Group Theory and Its Application to Physical Problems. Addison-Wesley Publ. Co., Inc., 1962.

3. Lomont, John S.: Applications of Finite Groups. Academic Press, 1959.

4. Harter, William George: Applications of Algebraic Representation Theory. PhD Thesis, Univ. of Calif., Irvine, 1967.

5. Bradley, C. J.: Space Groups and Selection Rules. J. Math. Phys., vol. 7, no. 7, July 1966, pp. 1145-1152.

NASA-Langley, 1968 —— 19 E-4329

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Notes, and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546